Poisson Process and the Exponential pdf

- N(t) is a point process that can represent the ***State*** of the system at time t.
- Goal: Find Prob [the system is in state k at t sec]=P(N(t)=k)=P[k,t]

  (if each increment in the process represents an arrival or "birth", then P[k,t]=Probability of # arrivals in t sec

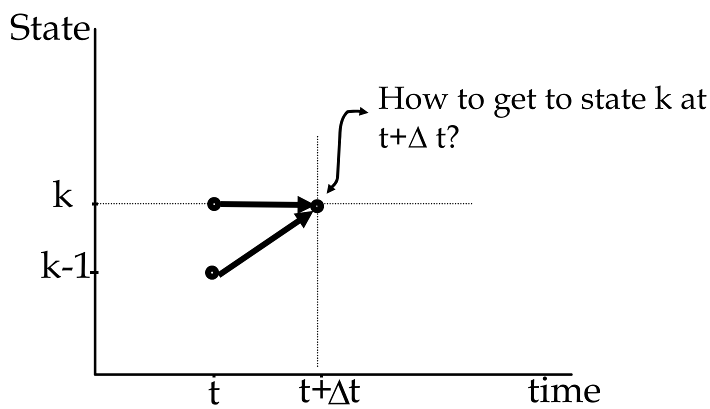
Analysis
Pure Birth (Poisson) Process: Assumptions

- Prob[1 arrivals in Δ t sec] = $\lambda \Delta t$
- Prob[0 arrivals in Δ t sec] = 1- $\lambda \Delta t$
- Independent Increments
- Number of arrivals in non-overlapping intervals of times are statistically independent random variables, i.e.,

  Prob [N arrivals in t, t+T AND M arrivals in t+T, t+T+τ]

  = Prob [N arrivals in t, t+T]*Prob[M arrivals in t+T, t+T+τ]

This is called a Poisson process or pure birth process

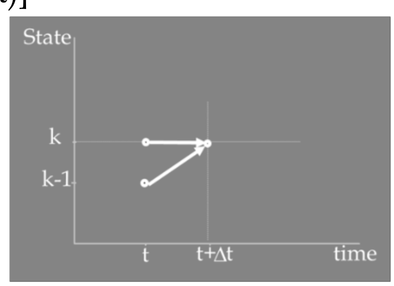


Analysis

- Define probability of k in the system at time t = Prob[k, t]
- Probability of k in the system at time t+ Δ t = Prob[k, t+ Δ t]

= Prob[k, t+ Δ t] Prob[(k in the system at time t and 0 arrivals in Δ t) or (k-1 in the system at time t and 1 arrival in Δ t)]

= (1- λ Δ t) Prob[k,t] + λ Δ t Prob[k-1,t]



Analysis

- Rearranging terms
  (Prob[k, t+ Δ t] - Prob[k,t])/ Δ t + λProb[k,t] = λ Prob[k-1,t]

- Letting Δ t --> 0 results in the following differential equation:

$$\frac{dProb[k,t]}{dt} + \lambda Prob[k,t] = \lambda Prob[k-1,t]$$

Analysis

- For k = 0 the solution is:
  - Prob[0,t]= $e^{-\lambda t}$
- For k = 1 the solution is:
  - Prob[1,t]= $\lambda t e^{-\lambda t}$
- For k = 2 the solution is:
  - Prob[2,t]= $\frac{(\lambda t)^2 e^{-\lambda t}}{2}$

Analysis

- In general the solution is a Poisson probability mass function of the form:

$$\text{Prob}\,[k\,,t] \;=\; \frac{(\lambda t)^{k}\, e^{-\lambda t}}{k!}$$

Analysis

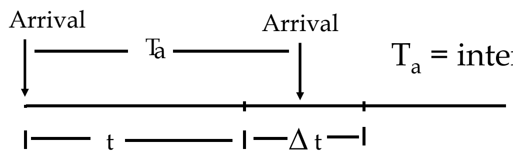
- A Poisson pmf of this from has the following moments:

$$E[k] = \lambda t$$

$$Var[k] = \lambda t$$

Poisson Arrival Process
The number of arrivals in any t second interval follows a Poisson probability mass function.

Interarrival Time Analysis



T_a = interarrival time λ Δ

Prob[t<Ta<t+ Δ t] = Prob[0 arrivals in t sec and 1 arrival in Δ t]
Prob[t<Ta<t+ Δ t] = Prob[k=0,t]Prob[k=1, Δ t]

Prob[t<Ta<t+ Δ t] = $(e^{-\lambda t})\lambda \Delta t e^{-\lambda \Delta t}$

Interarrival Time Analysis

Let Δ t → 0 results in the following

$$\text{Prob}[t < T_a < t + dt] = f_{T_a}(t)dt = \lambda e^{-\lambda t} dt$$

so

$$f_{T_a}(t) = \lambda e^{-\lambda t} \text{ for } t > 0 \quad f_{T_a}(t) = 0 \text{ for } t < 0$$

$$P[T_a < t] = u(t)(1 - e^{-\lambda t})$$
$$f_{T_a}(t) = u(t)\lambda e^{-\lambda t}$$

The distribution of interarrival times is exponential

Interarrival Time Analysis

The interarrival time for a Poisson arrival process follows an exponential probability density function with

$$E[T_a] = 1/\lambda \qquad Var[T_a] = 1/\lambda^2$$